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Fluctuations of Extensive Functions of Quenched Random Couplings

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An extensive quantity is a family of functions Ψ_V of random parameters, indexed by the finite regions V (subsets of $\mathbb{Z}^d$) over which Ψ_V are additive up to corrections satisfying the boundary estimate stated below. It is shown that unless the randomness is nonessential, in the sense that $\lim \Psi_V/|V|$ has a unique value in the absolute (i.e., not just probabilistic) sense, the variance of such a quantity grows as the volume of V. Of particular interest is the free energy of a system with random couplings; for such Ψ_V bounds are derived also for the generating function $E(e^{t\Psi})$. In a separate application, variance bounds are used for an inequality concerning the characteristic exponents of directed polymers in a random environment.

KEY WORDS: Random systems; static disorder; extensive quantities; fluctuations; directed polymers.

1. INTRODUCTION

Lattice models of statistical mechanics with random parameters in the interaction energy have been studied extensively in recent years. Among the best known examples are the random field Ising model (RFIM)[(1)] and the Edwards–Anderson spin-glass model.[(2)]

The Edwards–Anderson spin glass energy is

$$H = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \tag{1.1}$$

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where J_{ij} are random couplings and $\sigma_i = \pm 1$ are the spin variables. In the RFIM case

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i \eta_i \sigma_i \tag{1.2}$$

with random fields η_i (see Section 5 for precise definitions). In these models the finite-volume free energies F_V are random variables, since they depend on the random parameters $(\{J_{ij}\}, \{\eta_i\})$ entering H. This paper focuses on the fluctuations of such functions of the randomness, discussed here within the context of *extensive quantities*, of which the free energies and the ground-state energies are prime examples.

Before stating the results, let us note that in the Sherrington–Kirkpatrick ("mean-field") version of the spin-glass model one finds that the variance of the free energy of an N-particle system can behave in different ways, being of the order $O(N)$ at low temperatures and of the order $O(1)$ at high temperatures.[3]

In contrast, the general results derived here show that in the finite-dimensional models mentioned above the variance of the free energy, as well as the variance of the ground-state energy, are always of the order of the volume, i.e.,

$$k\,|V| \leqslant \mathrm{Var}(F_V) \leqslant K\,|V| \tag{1.3}$$

with volume-independent constants $0 < k < K < \infty$.

The variance of a random variable X is defined as $\mathrm{Var}(X) = E[(X - E(X))^2]$, with E denoting expected value.

The main result of this paper is a *lower bound* (the upper bound being quite elementary) on the variance of *extensive quantities* Ψ_V (see Definition 2.1). We show that for each such quantity Ψ

$$\mathrm{Var}(\Psi_V) \geqslant k\,|V|, \qquad k = k_\Psi > 0 \tag{1.4}$$

(for sufficiently "regular" regions V), *unless* the randomness is irrelevant for Ψ in the sense that the limiting density

$$\lim_{\substack{V = [-L, L]^d \\ L \to \infty}} \frac{\Psi_V}{|V|} \tag{1.5}$$

is constant in an absolute (not just probabilistic) sense.

In view of this general statement, one could ask whether it is also true that the fluctuations of an extensive quantity are Gaussian, on the scale of

$\sqrt{|V|}$. In Section 5 we show that this need not be the case—though the counterexamples show only very mild violations of the proposed rule.

Fluctuation bounds have a number of applications. Results related to those discussed here were instrumental in the proof of rounding effects of the quenched randomness on first-order phase transitions in low-dimensional systems.[4] Another application, presented in Section 6, is an inequality for characteristic exponents of the model of directed polymers in a random environment.

The paper is organized as follows. Section 2 includes the definition of the extensive quantities and the statement of the main result, which is proven in Section 4. In Section 3 we present some elementary, though useful, bounds on the fluctuations of quantities depending on independent random variables. Section 5 discusses applications of the general bounds to lattice models of disordered systems. Finally, Section 6 contains an application of the variance bounds of Section 2 to a first passage problem.

2. VARIANCE BOUNDS

In this section we define our notion of an extensive quantity and state the general result. Functions of random parameters associated with a lattice are discussed here in an abstract setting—with no underlying spin model.

Let $\{\eta_\alpha\}$ be a family of independent random variables, with α ranging over a collection of finite subsets of the lattice $\mathbb{Z}^d$, satisfying

$$E(\eta_\alpha)=0; \qquad E(\eta_\alpha^2)=1 \tag{2.1}$$

The joint distribution of the $\{\eta_\alpha\}$ as well as the distribution of an individual η_α will be denoted by ν. We assume that ν is translation invariant.

Definition 2.1. An extensive quantity is a family of functions $\Psi=\{\Psi_V\}$ indexed by the finite subsets of $\mathbb{Z}^d$ which satisfy:

(i) Each Ψ_V is a continuous function of $\{\eta_\alpha \mid \alpha\subset V\}$.

(ii) For every finite set V and a cubic region $\Lambda\subset V$,

$$|\Psi_V(\eta)-[\Psi_{V\setminus\Lambda}(\eta)+\Psi_\Lambda(\eta)]|\leqslant B_\Lambda(\eta_{\mathscr{D}(\Lambda)}) \tag{2.2}$$

where B_Λ depends only on η in the boundary set $\mathscr{D}(\Lambda)=\{\alpha\subset\mathbb{Z}^d \mid \alpha\cap\Lambda\neq\varnothing \text{ and } \Lambda^c\neq\varnothing\}$ and obeys the bounds

$$E(B_\Lambda^2)<\infty \tag{2.3}$$

and, for Λ a translate of $[-L, L]^d$,

$$E(B_\Lambda) = o(L^d) \qquad \text{as} \quad L \to \infty \tag{2.4}$$

An extensive quantity is called translation invariant if for all η and x, $\Psi_{T_x V}(T_x \eta) = \Psi_V(\eta)$, where T_x is the translation by the vector x.

As a preliminary remark we note that extensive quantities obey a "law of large numbers" in the following sense.

Proposition 2.2. Let Ψ be a translation-invariant extensive quantity. If the distribution of η is translation invariant and ergodic, then for $V \to \mathbb{Z}^d$ in the sense of van Hove,[3] $\Psi_V(\eta)/|V|$ converges almost surely to a constant ($|V|$ being the number of the lattice sites in V).

Properties of this type are generally known (see refs. 6). The proposition applies in particular in the case of independent variables considered here. Note that for periodic configurations η the limit

$$\bar{\Psi}(\eta) = \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} \Psi_V(\eta) \tag{2.5}$$

(which for extensive quantities always exists) may in general depend on η.

Our main result, proven in Section 4, concerns the order of fluctuations of $\Psi_V(\eta)$:

Theorem 2.3. For a translation-invariant extensive quantity Ψ, let $I_\Psi = \{\bar{\Psi}(\eta) \,|\, \eta \text{ a periodic configuration}\}$, where $\bar{\Psi}(\eta)$ is defined by (2.5). If I_Ψ contains more than one point, then there exists $T < \infty$ (large enough) and $k > 0$ with which, for any finite $V < \mathbb{Z}^d$,

$$\operatorname{Var}(\Psi_V) \geqslant k \, |\operatorname{Int}_T V| \tag{2.6}$$

where $\operatorname{Int}_T V$ is the interior of V:

$$\operatorname{Int}_T V = \{x \in V \,|\, \operatorname{dist}(x, V^c) \geqslant T\} \tag{2.7}$$

Furthermore, regardless of the above assumption about I_Ψ but under the finite-range assumption, that for some R all η_α with $\operatorname{diam}(\alpha) > R$ are constant with probability 1, there exists a $K < \infty$ such that for every finite V,

$$\operatorname{Var}(\Psi_V) \leqslant K \, |V| \tag{2.8}$$

Remarks. 1. For V sufficiently regular the lower bound (2.6) is of the order of the volume of V. More precisely: if $V_n \to \mathbb{Z}^d$ in the sense of van Hove, then $\lim_{n \to \infty}(|\operatorname{Int}_T V_n|/|V_n|) = 1$ for any T.

2. Further information on the probability of large fluctuations of the finite-volume free energies is provided by a bound on the moment generating function presented in Section 5.

3. GENERAL FLUCTUATION BOUNDS

In the derivations of the above and other results, use is made of general bounds on fluctuations of random variables of the form

$$X(\tau_1,\dots,\tau_N) \tag{3.1}$$

where $\tau_1,\dots,\tau_N$ are independent random variables with values in a measurable space $\mathscr{E}$ and X is a real-valued measurable function on $\mathscr{E}^N$.

3.1. Variance Bounds

Let ρ_i denote the probability measures on $\mathscr{E}$ describing the distributions of τ_i. Since τ_j, $j=1,\dots,N$, are independent, the conditional expectation of X conditioned on the event $\{\tau_i=t\}$ is given by

$$E(X\,|\,\tau_i=t)=\int \prod_{j(\neq i)} \rho_j(d\tau_j)\, X(\tau_1,\dots,\tau_{i-1},t,\tau_{i+1},\dots,\tau_N) \tag{3.2}$$

For X as in (3.1) we denote by $\widehat{\mathrm{Var}}_i(X)$ and $\widecheck{\mathrm{Var}}_i(X)$ the following "localized" variances:

$$\begin{aligned}
\widehat{\mathrm{Var}}_i(X) &\stackrel{\text{def}}{=} \mathrm{Var}(E(X\,|\,\tau_i))\\
&=\int \rho_i(dt)\Bigg[\int \prod_{j(\neq i)} \rho_j(d\tau_j)\, X(\tau_1,\dots,\tau_i=t,\dots,\tau_N)\\
&\qquad -\int \prod_{j=1}^{N} \rho_j(d\tau_j)\, X(\tau_1,\dots,\tau_N)\Bigg]^2
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\widecheck{\mathrm{Var}}_i(X) &\stackrel{\text{def}}{=} \mathrm{Var}(X-E(X\,|\,\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_N))\\
&=\int \prod_{j=1}^{N} \rho_j(d\tau_j)\left[X(\tau_1,\dots,\tau_N)-\int \rho_i(dt)\, X(\tau_1,\dots,\tau_i=t,\dots,\tau_N)\right]^2
\end{aligned} \tag{3.4}$$

Proposition 3.1. For any X of the form (3.1),

$$\sum_{i=1}^{N} \widehat{\mathrm{Var}}_i(X) \leqslant \mathrm{Var}(X) \leqslant \sum_{i=1}^{N} \widecheck{\mathrm{Var}}_i(X) \tag{3.5}$$

Proof. The argument is conveniently carried in the Hilbert space

$$\mathscr{H} = L^2\left(\mathscr{E}^N, \prod_{i=1}^{N} \rho_i(d\tau_i)\right) \tag{3.6}$$

For $i = 1,..., N$ let R_i be the operator in $\mathscr{H}$ defined by

$$(R_i Y)(\tau_1,..., \tau_N) = \int \rho_i(dt)\ Y(\tau_1,..., \tau_{i-1}, t, \tau_{i+1},..., \tau_N) \tag{3.7}$$

and let

$$P_i = \prod_{j=1}^{i} R_j; \qquad P_0 = I \tag{3.8}$$

These R_i and P_i $(i = 1,..., N)$ form a family of commuting orthogonal projections. With the natural identification of random variables of the form (3.1) and their various conditional expectations with elements of $\mathscr{H}$ we have

$$\begin{aligned} X &= P_0 X \\ E(X) &= P_N X \\ E(X \mid \tau_i) &= \prod_{j \neq i} R_j X \\ \widehat{\mathrm{Var}}_i(X) &= \left\| \left(\prod_{j \neq i} R_j \right) (I - R_i) X \right\|_2^2 \\ \widecheck{\mathrm{Var}}_i(X) &= \|(I - R_i) X\|_2^2 \end{aligned} \tag{3.9}$$

Note that the random variables corresponding to the vectors $P_i X$ form a martingale (with respect to the σ-algebras $\mathscr{F}_i$ generated by $\{\tau_1,..., \tau_i\}$) and therefore the vectors $(P_i X - P_{i-1} X)$, being martingale differences, are mutually orthogonal. The last assertion can also be verified by simple algebra.

It follows that

$$\mathrm{Var}(X) = \|X - E(X)\|_2^2 = \sum_{i=1}^{N} \|(P_{i-1} - P_i) X\|_2^2 = \sum_{i=1}^{N} \|P_{i-1}(I - R_i) X\|_2^2 \tag{3.10}$$

and consequently, using the representation (3.9),

$$\begin{aligned} \sum_{i=1}^{N} \widehat{\mathrm{Var}}_i(X) &= \sum_{i=1}^{N} \left\| \left(\prod_{j(\neq i)} R_j \right) (I - R_i) X \right\|_2^2 \leqslant \mathrm{Var}(X) \\ &\leqslant \sum_{i=1}^{N} \|(I - R_i) X\|_2^2 = \sum_{i=1}^{N} \widecheck{\mathrm{Var}}_i(X) \quad \blacksquare \end{aligned} \tag{3.11}$$

3.2. An Upper Bound for the Moment Generating Function

Useful information about the nature of fluctuations is often conveyed by the moment generating function. In Section 5 we make use of the following bound.

Proposition 3.2. Let $\tau_1,..., \tau_N$ be independent real random variables with

$$E(e^{t\tau_i}) \leqslant e^{c_i t^2} \tag{3.12}$$

for all $t \in \mathbb{R}$. For a function $X(\tau_1,..., \tau_N)$ let L_i be the Lipschitz constants:

$$L_i = \sup_{\tau_1,...,\tau_N, \Delta\tau} \{|X(\tau_1,..., \tau_i + \Delta\tau,..., \tau_N) - X(\tau_1,..., \tau_i,..., \tau_N)|/|\Delta\tau|\} \tag{3.13}$$

Then, for every real t,

$$E(\exp\{t[X - E(X)]\}) \leqslant \exp\left(2 \sum_{k=1}^{N} c_k L_k^2 t^2\right) \tag{3.14}$$

Proof. 1. Let us prove (3.14) first for $N = 1$, i.e., for functions of a single variable. In this case we omit all the indices. Let ν be the distribution of τ. Introducing a second, independent copy of τ—denoted by τ'—we have

$$\begin{aligned}
&E(\exp\{t[X - E(X)]\}) \\
&\quad = \int \nu(d\tau) \exp\left\{t \int \nu(d\tau')\, [X(\tau) - X(\tau')]\right\} \\
&\quad \leqslant \int \nu(d\tau) \int \nu(d\tau') \exp\{t[X(\tau) - X(\tau')]\} \\
&\quad = \iint \nu(d\tau)\, \nu(d\tau') \cosh\{t[X(\tau) - X(\tau')]\} \\
&\quad = \iint \nu(d\tau)\, \nu(d\tau') \cosh[tL(\tau - \tau')] \\
&\quad = \iint \nu(d\tau)\, \nu(d\tau')\, \tfrac{1}{2}\{\exp[tL(\tau - \tau')] + \exp[tL(\tau' - \tau)]\} \\
&\quad \leqslant \exp(2cL^2t^2)
\end{aligned} \tag{3.15}$$

In the above sequence of steps, we applied the Jensen inequality and then symmetrized the expression in order to make a better use of the Lipschitz condition (3.13).

2. For general $N \geqslant 1$

$$
\begin{aligned}
&E(\exp\{t[X-E(X)]\}) \\
&\quad = \frac{\int \prod_{k=1}^{N} \nu(d\tau_k) \exp[tX(\tau_1,..., \tau_N)]}{[t \int \prod_{k=1}^{N} \nu(d\tau_k)\, X(\tau_1,..., \tau_N)]} \\
&\quad = \prod_{k=1}^{N} \frac{\int \prod_{i=1}^{k} \nu(d\tau_i) \exp[t \int \prod_{i=k+1}^{N} \nu(d\tau_i)\, X(\tau_1,..., \tau_N)]}{\int \prod_{i=1}^{k-1} \nu(d\tau_i) \exp[t \int \prod_{i=k}^{N} \nu(d\tau_i)\, X(\tau_1,..., \tau_N)]} \\
&\quad \leqslant \prod_{k=1}^{N} \sup_{\tau_1,...,\tau_{k-1}} \frac{\int \nu(d\tau_k) \exp[t\tilde{X}(\tau_1,..., \tau_k)]}{\exp[t \int \nu(d\tau_k)\, \tilde{X}(\tau_1,..., \tau_k)]}
\end{aligned}
\tag{3.16}
$$

where $\tilde{X}(\tau_1,..., \tau_k) = \int \prod_{i=k+1}^{N} \nu(d\tau_i)\, X(\tau_1,..., \tau_N)$.

Since, for any choice of $\tau_1,..., \tau_{k-1}$, $\tilde{X}$ as a function of τ_k satisfies the Lipschitz condition with the constant L_k, the proposition follows from the case $N=1$ proven above. ∎

4. VARIANCE BOUNDS FOR EXTENSIVE QUANTITIES

For the general result stated in Theorem 2.3 we apply Proposition 3.1 at the level of block variables, as seen in the proof of the following lemma.

Lemma 4.1. Let Ψ be an extensive quantity. If for some rectangular region Λ the collection $\mathscr{G}$ of pairs of coupling configurations (η', η'') satisfying

$$\Psi_\Lambda(\eta') - \Psi_\Lambda(\eta'') \geqslant \delta |\Lambda| \tag{4.1}$$

with some $\delta > 2E(B_\Lambda)/|\Lambda|$ is of positive $\nu \otimes \nu$ measure, then, for every finite $V \subset \mathbb{Z}^d$,

$$\operatorname{Var}(\Psi_V) \geqslant \tfrac{1}{2}[\delta - 2E(B_\Lambda)/|\Lambda|]^2\, m\, |\Lambda|\, |V|_\Lambda \tag{4.2}$$

with $m = \nu \otimes \nu(\mathscr{G})$. Here $|V|_\Lambda$ denotes the maximal volume of a subset of V tiled by disjoint translates of Λ.

Proof. Let $\Lambda_1,..., \Lambda_M$ be a collection of disjoint translates of Λ contained in V. We shall apply Proposition 3.1 to Ψ_V conceived as

$$\Psi_V = X(\tau_1,..., \tau_M,..., \tau_N) \tag{4.3}$$

where for $i = 1,...., M$, τ_i represent the block variables

$$\tau_i \equiv \eta_{(\Lambda_i)} = \{\eta_\alpha \mid \alpha \subset \Lambda_i\} \tag{4.4}$$

and for $i = M+1, \ldots, N$, τ_i varies over all the other variables η_α on which Ψ_V may depend.

For $i \leqslant M$ we have, using the decomposition $\Psi_V = \Psi_A + \Psi_{V\setminus A} + (\Psi_V - \Psi_A - \Psi_{V\setminus A})$ and estimating the "boundary term" by B_A of (2.2), which does not depend on η_A,

$$\widehat{\mathrm{Var}}_i(X) = \frac{1}{2}\int \nu(dt)\,\nu(dt')\,[E(X\,|\,\tau_i = t) - E(X\,|\,\tau_i = t')]^2$$

$$\geqslant \frac{1}{2}\int_{\mathscr{G}} \nu(d\eta_{A_i})\,\nu(d\eta'_{A_i})\,[\Psi_{A_i}(\eta) - \Psi_{A_i}(\eta') - 2E(B_{A_i})]^2$$

$$\geqslant \frac{1}{2} m[\delta\,|A| - 2E(B_{A_i})]^2 \tag{4.5}$$

Proposition 3.1 implies now

$$\mathrm{Var}(\Psi_V) \geqslant \frac{1}{2}[\delta - 2E(B_A)/|A|]^2 \sum_{i:A_i \subset V} m\,|A|^2 \tag{4.6}$$

from which (4.2) readily follows. ∎

Proof of Theorem 2.3. (i) The lower bound. It follows from our assumptions on I_Ψ that there is some $\delta > 0$ such that in every cube A, with edge length L sufficiently big, there is a pair of coupling configurations $\{\eta_A, \eta'_A\}$ with

$$\Psi_A(\eta_A) - \Psi_A(\eta'_A) > \delta\,|A| \tag{4.7}$$

The continuity of Ψ_A implies that for each such L the inequality (4.7) holds for $\{\eta_A, \eta'_A\}$ in a set of positive $\nu \otimes \nu$ measure (depending on L). Let us choose L big enough so that

$$E(B_A)/|A| < 2\delta \tag{4.8}$$

[which by (2.4) holds for L large enough]. A simple covering argument shows that $|V|_A \geqslant \mathrm{const} \cdot \mathrm{Int}_L V$, and hence the claimed lower bound (2.6) readily follows from the conclusion of Lemma 4.1.

(ii) For the upper bound we apply Proposition 3.1 with η_α, $\alpha \subset V$, in the role of τ_i. In order to estimate $\widecheck{\mathrm{Var}}_\alpha\, \Psi_V$, we choose for A a rectangle containing α. The corresponding decomposition of Ψ_V, as above (4.5), shows that the variance of Ψ_V as a function of the particular η_α, with the other random couplings fixed (at some $\{\eta_\beta\}$), is not larger than

$$\frac{1}{2}\int \nu(d\eta_\alpha)\,\nu(d\eta'_\alpha)\,[\Psi_\alpha(\eta_\alpha, \{\eta_\beta\,|\,\alpha \neq \beta\}) - \Psi_\alpha(\eta'_\alpha, \{\eta_\beta\,|\,\alpha \neq \beta\}) + 2B_A(\eta_{\mathscr{D}(A)})]^2 \tag{4.9}$$

Integrating over η_β with $\beta \neq \alpha$, we obtain (via simple applications of the Schwarz inequality)

$$\widecheck{\mathrm{Var}}_\alpha\, \Psi_V \leqslant \mathrm{Var}(\Psi_A) + 2E(B_A^2) \tag{4.10}$$

The above quantity is finite by the assumption (2.3). In the finite-range case, the summation of (4.10) over α yields an upper bound on Var Ψ_V which is proportional to $|V|$, as claimed in (2.9). ▮

5. FREE ENERGY FLUCTUATIONS IN RANDOM SYSTEMS OF STATISTICAL MECHANICS

The above results have implications for the free energy in models with static disorder (quenched randomness). We consider a general class of spin models on $\mathbb{Z}^d$ with interactions of the form

$$H(\sigma) = H_0(\sigma) + \sum_\alpha (h_\alpha + \varepsilon_\alpha \eta_\alpha)\, g_\alpha(\sigma) \tag{5.1}$$

where α ranges over finite subsets of the lattice and $\{\eta_\alpha\}$ is a family of independent random variables with a translation-invariant distribution, satisfying the normalization (2.1). H_0 is a nonrandom component of the interaction. It may be given by an expression of the form

$$H_0(\sigma) = \sum_{A \subset \mathbb{Z}^d} \Phi_A(\sigma) \tag{5.2}$$

where the summation is over finite A and each Φ_A is a bounded function of the spins in A. In the sum in (5.1), for each α, g_α depends only on the spin variables $\{\sigma_i\}_{i \in \alpha}$, and is assumed to be uniformly bounded by 1. Its coefficient consists of two terms—the deterministic h_α and the random $\varepsilon_\alpha \eta_\alpha$.

We assume that the system is translation invariant, in the sense of strict invariance for the nonrandom part and stochastic invariance for the random component of $H(\sigma)$; i.e., for every lattice translation T, $\Phi_{TA}(\cdot) = \Phi_A(T\cdot)$, $g_{T\alpha}(\cdot) = g_\alpha(T\cdot)$, $h_{T\alpha} = h_\alpha$, $\varepsilon_{T\alpha} = \varepsilon_\alpha$, and, furthermore, $\{\eta_\alpha\}$ have a translation-invariant distribution.

The finite-volume free energy of the model at the inverse temperature β equals

$$F_V(\eta; \beta, \varepsilon) = -\frac{1}{\beta} \log \int \rho_0(d\sigma)\, e^{-\beta H_V(\sigma)} \tag{5.3}$$

where ρ_0 is the *a priori* probability distribution of the spin variables, and

H_V is the partial sum, consisting of the terms involving the spin in V. For $\beta = \infty$ (zero temperature), F_V is the ground-state energy:

$$F_V(\eta;\, \infty, \varepsilon) = \inf_\sigma H_V(\sigma) \tag{5.4}$$

Remark. In the above definition, $H_V(\sigma)$ consists of those interaction terms which involve only the spins in V. There is a certain amount of non-uniqueness in this notion, since the infinite-volume interactions have more than one representation in terms of the local sums (5.2); alternatively said, finite-volume free energies typically depend on the boundary conditions. Curiously, that effect (whose magnitude is proportional to $|\partial V|$) does not ruin the fact derived here that the fluctuations of F_V are only of the order of $|V|^{1/2}$.

Proposition 5.1. If H_0 is of the form (5.2), with

$$\sum_{\alpha \subset \mathbb{Z}^d, \alpha \ni 0} \frac{|h_\alpha| + \varepsilon_\alpha}{|\alpha|} < \infty \tag{5.5}$$

and

$$\sum_{A \subset \mathbb{Z}^d, A \ni 0} \frac{\|\Phi_A\|_\infty}{|A|} < \infty \tag{5.6}$$

($\|\cdot\|_\infty$ denoting the supremum norm), then for each $\beta \leqslant \infty$ the quantity $\{F_V\}$ defined above is extensive.

Proof. Using the inequality

$$\left|\log \int \rho_0(d\sigma)\, e^{A(\sigma) + B(\sigma)} - \log \int \rho_0(d\sigma)\, e^{A(\sigma)}\right| \leqslant \|B\|_\infty \tag{5.7}$$

we have, for each η

$$\begin{aligned} &|F_V(\eta) - F_\Lambda(\eta) - F_{V\setminus\Lambda}(\eta)| \\ &\quad\leqslant \sup_\sigma |H_V(\sigma) - H_\Lambda(\sigma) - H_{V\setminus\Lambda}(\sigma)| \\ &\quad\leqslant \sum_{A \in \mathscr{D}(\Lambda)} \|\Phi_A\|_\infty + \sum_{\alpha \in \mathscr{D}(\Lambda)} (|h_\alpha| + \varepsilon_\alpha |\eta_\alpha|) \stackrel{\text{def}}{=} B_\Lambda(\eta) \end{aligned} \tag{5.8}$$

The last expression depends only on $\eta_{\mathscr{D}(\Lambda)}$. Integrating it over η, we obtain

$$\begin{aligned} E(B_\Lambda) &\leqslant \sum_{x \in \Lambda} \sum_{\substack{A \ni x,\\ A \in \mathscr{D}(\Lambda)}} \frac{\|\Phi_A\|_\infty}{|A|} + \sum_{x \in \Lambda} \sum_{\substack{\alpha \ni x,\\ \alpha \in \mathscr{D}(\Lambda)}} \frac{|h_\alpha| + \varepsilon_\alpha}{|\alpha|} \\ &\leqslant \sum_{x \in \Lambda} \sum_{\substack{A \ni 0,\\ \operatorname{diam}(A) \geqslant d(x, \partial\Lambda)}} \frac{\|\Phi_A\|_\infty + |h_\alpha| + \varepsilon_A}{|A|} = o(|\Lambda|) \end{aligned} \tag{5.9}$$

where the last conclusion is by (5.5) and (5.6). This verifies (2.4). The second-moment condition (2.3) follows from (2.1). ▮

Specific examples are the following models with spins taking values ± 1.

(i) Random-Field Ising Model (RFIM).[1] The RFIM interaction is

$$H(\sigma) = -\sum_{x,\,y} J_{x-y}\sigma_x\sigma_y - \sum_x \varepsilon\eta_x\sigma_x \tag{5.10}$$

Here J_z are ferromagnetic (i.e., positive) coupling constants of finite range, and x and y run through $\mathbb{Z}^d$.

(ii) The Edwards–Anderson Spin-Glass Model.[2] In this case

$$H(\sigma) = -\sum_{|x-y|=1} \eta_{x,\,y}\sigma_x\sigma_y \tag{5.11}$$

with the i.i.d. random variables $\eta_{x,\,y}$ (of a symmetric distribution). The parameter ε is omited, since its role is played in this case by the inverse temperature β.

5.1. Upper Bounds

In view of Proposition 5.1, the general results on extensive quantities apply to free energies. The upper bound of Theorem 2.3 yields the following:

Proposition 5.2. If the interaction (5.1) is of finite range, then there exists a constant K (dependent only on the interaction) such that for every $\beta \leqslant \infty$ and for every finite set V

$$\mathrm{Var}(F_V) \leqslant K\,|V| \tag{5.12}$$

Further information on the probability of large fluctuations is contained in the following consequence of Proposition 3.2.

Proposition 5.3. Let H be as above; then for every $\beta \leqslant \infty$,

$$\frac{E(\exp[tF_V])}{\exp[tE(F_V)]} \leqslant \exp\left(2\sum_{\alpha \ni 0} C_\alpha \varepsilon_\alpha^2\,|V|\,t^2\right) \qquad \text{for all} \quad t \in \mathbb{R} \tag{5.13}$$

with the constants C_α defined by the conditions $E(\exp(t\eta_\alpha)) \leqslant \exp(C_\alpha t^2)$ (for all $t \in \mathbb{R}$).

The moment generating function bound (5.13) implies the variance bound (5.12) with

$$K = 4 \sum_{\alpha \ni 0} C_\alpha \varepsilon_\alpha^2 \tag{5.14}$$

Furthermore, by a standard application of the exponential Chebyshev inequality, (5.13) yields the following bound on the probabilities of deviations of F_V.

Corollary 5.4.

$$\mathrm{Prob}(|F_V| \geqslant b) \geqslant \exp(-b^2/4K\,|V|) \tag{5.15}$$

with K given by (5.14).

Remark. In the study of the RFIM it is important to analyze probability bounds of the type (5.15) for the differences of the finite-volume free energies, with + and − boundary conditions,[7]

$$G_V = F_{V,+} - F_{V,-} \tag{5.16}$$

It is easy to see that the above corollary applies to this quantity as well, thus proving a particular case ($B \neq \varnothing$) of the conjecture stated in ref. 7, that in $d \geqslant 3$ dimensions

$$\mathrm{Prob}(|G_A - G_B| \geqslant b) \leqslant \exp\left[-\frac{\mathrm{const}\cdot b^2}{\varepsilon^2(|A\backslash B| + |B\backslash A|)}\right] \tag{5.17}$$

for every two finite sets $A, B \subset \mathbb{Z}^d$. The full statement is still unproven. In ref. 7 it is shown that such a result would suffice for a proof of existence of the phase transition (seen by varying h) in the three-dimensional RFIM. The existence of the phase transition was proven by other means in refs. 8 (for $T = 0$) and 9 (for $T > 0$, small enough).

5.2. Lower Bounds

In order to apply Proposition 2.2 to a particular model, one has to prove existence of two periodic configurations of the random parameters for which the limit

$$\lim_{V \to \infty} \frac{F_V}{|V|} \tag{5.18}$$

takes two different values. We do this in detail only for the RFIM case, and under the following additional assumption:

(A) The restriction of ν to at least one of the sets $R_+\setminus\{0\}$ and $R_-\setminus\{0\}$ is supported in more than one point.

We describe below how this assumption can be avoided.

Let (A) be satisfied and let $\varepsilon > 0$. For the two periodic configurations we choose the constant configurations $\eta_i \equiv \eta^{(1)}$ and $\eta_i \equiv \eta^{(2)}$, with $\eta^{(1)}$, $\eta^{(2)} \in \operatorname{supp} \nu$ and (say) $\eta^{(1)} > \eta^{(2)} > 0$. The difference of the finite-volume free energies corresponding to these two field configurations satisfies

$$\frac{F_V(\eta^{(1)}) - F_V(\eta^{(2)})}{|V|} \geqslant \varepsilon(\eta^{(1)} - \eta^{(2)}) \frac{1}{|V|} \sum_{i \in V} \langle \sigma_i \rangle_{V, \eta^{(2)}}$$
$$\geqslant \varepsilon(\eta^{(1)} - \eta^{(2)}) \langle \sigma_0 \rangle_{\{0\}, \eta^{(2)}} > 0 \tag{5.19}$$

where $\langle \cdot \rangle_{\Lambda, \eta}$ is the expectation in the Gibbs distribution for the (nonrandom) Ising model at the temperature β^{-1} and uniform magnetic field η, while $\langle \sigma_0 \rangle_{\{0\}, \eta^{(2)}}$ (>0) is the magnetization in the one-particle system (the second step in (5.19) is by the second Griffiths inequality[10]). Hence, the assumptions of the Proposition 2.2 are met, and we have:

Proposition 5.5. For each RFIM with $\varepsilon > 0$, satisfying the condition (A) stated above, there exist some $k > 0$ and $L > 0$ such that for every finite V [with $\operatorname{Int}_L V$ defined below (2.7)]

$$\operatorname{Var}(F_V) \geqslant k\, |\operatorname{Int}_L V| \tag{5.20}$$

For completeness, let us outline here an argument which permits one to remove the clause (A) seen above. The remaining case is when the distribution of the random parameters is supported at two points with opposite signs, say η_1 and η_2. To generate a pair of periodic field configurations with different values for the limit (5.18), one can choose the constant configuration $\eta \equiv \eta_1$ and its modification obtained by replacing η_1 by η_2 on a periodic sublattice of low density. In order to prove the desired statement—which intuitively is obvious—one can use an extension of the Fortuin–Kasteleyn representation.[11,12] In that representation the ratio of the partition functions for systems in a finite volume with the two magnetic field configurations is equal to the probability, calculated in a bond percolation model, of the event that the bond configuration carries no frustration. This probability is bounded above by the probability of an intersection of local events whose number is proportional to the volume of the system. While these events are not independent, they are sufficiently noncoercive so that the conditional probability of each one of them conditioned on the status of all the others is less than a uniform constant, smaller than 1. The ratio of the partition functions decays exponentially, and hence the corresponding free energy densities are unequal.

Using related arguments, one can prove an inequality like (5.20) for a variety of other models, including *spin glasses* (for which the proof of the existence of fluctuations in the free energy density may be aided by the Mattis construction of tractable periodic interactions).

Remarks. 1. A question naturally suggested by the results presented above is whether the finite-volume free energy satisfies a central limit theorem, in the sense that the distribution of $F_V - E(F_V)/\sqrt{|V|}$ has a normal limit. We expect such a general statement to be false, though the required modification may be minor, due to a mechanism seen in a caricature of the RFIM's ground state. In this caricature we associate to each random field configuration two spin configurations corresponding to $+$ and $-$ boundary conditions. In the "+" configuration, $\sigma_x = +1$ except when $\eta_x < -2d$; and in the "$-$" configuration $\sigma_x = -1$ except when $\eta_x > 2d$. The energy of each configuration is given by the RFIM interaction (5.10) with $\varepsilon = 1$. We define $\psi_V(\eta)$ as the *minimum* of the two energies. A standard central limit theorem *does* apply to the energies of the two configurations. Consequently, the limiting distribution for the (scaled) ground-state energy with free boundary conditions is that of a minimum of a pair of jointly Gaussian random variables, which are dependent but not identical. That distribution is, of course, (mildly) non-Gaussian.

2. One may expect the above caricature to offer an approximation to the 3D RFIM at low temperatures and for small disorder, where large contours are sparse (see refs. 8 and 9). An even stronger statement may be true—in the context of disordered systems with finitely many "phases," which are generated by deterministic boundary conditions. The q-state Potts model in an appropriately defined random magnetic field may be an example of such a system. We expect that for such a model the distributional limit to $F_V - E(F_V)/\sqrt{|V|}$ is either Gaussian or the minimum of a finite collection of (jointly) Gaussian variables, depending on whether F_V is computed for a "pure phase" or for a mixture of phases.

3. There is a natural notion of finite-volume free energy associated with a Gibbs state for the given interaction (such quantities were found very useful in ref. 4). Its definition is

$$F_{V,\rho}(\beta) = \frac{1}{\beta} \log \rho(e^{+\beta H_V}) \tag{5.21}$$

where H_V is as above, ρ is an infinite-volume Gibbs state, and the sign in the exponent is chosen so as to produce a partial cancellation with the corresponding term in the Gibbs factor. In systems with random parameters, Gibbs measures depend on η. In ref. 4 we show that if the infinite-volume Gibbs state is chosen so as to vary covariantly with η (under translations and local changes), then a partially averaged version of $F_{V,\rho}$,

$$\tilde{F}_{V,\rho} \stackrel{\text{def}}{=} \int \nu(d\eta_{V^c})\, F_{V,\rho_\eta}$$

obeys a central limit theorem. (Such a covariant family $\{\rho_\eta\}$ is associated with a nonrandomly defined phase. It is known to exist for the ferromagnetic RFIM, but whether it exists in the general case is an interesting unresolved issue.)

6. AN INEQUALITY FOR CHARACTERISTIC EXPONENTS OF DIRECTED POLYMERS IN A RANDOM ENVIRONMENT

In optimization problems of the "first passage" type, the fluctuations in the value of the optimized quantity are related to the size of "typical excursions." We exhibit here such a relation in the context of a model of directed polymers, various aspects of which have been studied in refs. 13–15.

Let ω represent a nearest-neighbor walk on $\mathbb{Z}^d$ starting from the origin, and let $\{\eta(t, x) \mid x \in \mathbb{Z}^d;\ t = 1, 2,...\}$ be a family of independent, identically distributed, random variables with a common distribution $\nu(d\eta)$. With each walk of length T we associate the quantity

$$H_T(\omega; \eta) = \sum_{t=1}^{T} \eta(t, \omega(t)) \tag{6.1}$$

and to each "environment" η we associate

$$F_T(\eta) = \inf_{\omega} H_T(\omega; \eta) \tag{6.2}$$

The space-time graph of a walk can be though of as a "directed polymer" in $d+1$ dimensions. In this case H_T has the interpretation of the energy of a polymer in the random environment described by the random variables $\{\eta(t, x)\}$, and F_T becomes the ground-state energy. Another motivation for this model is "first-passage" problems (see ref. 16), for which $\eta(t, x)$ play the role of local passage times [and t indicates just a preferred direction in the $(d+1)$-dimensional space]. In that case, F_T is the minimal passage time among walks from the origin to the hyperplane $\{(t, x) \mid t = T\}$.

The variables $\eta(t, x)$ are assumed here to have an absolutely continuous distribution of bounded density:

$$\left|\frac{\partial v}{\partial \eta(t, x)}\right| \leqslant K \tag{6.3}$$

In particular, it follows from the continuity of v that with probability 1 for each T there exists a *unique* energy-minimizing polymer ω_T of length T. (No assumptions are required for the mere existence of a minimum in a "first passage" problem with a specified starting point and a specified number of steps.)

The mean displacement and the fluctuations of the ground-state energy are expected to obey power laws[13] with characteristic exponents denoted by ζ and χ:

$$\operatorname{Var}(|\omega_T(T)|) \sim T^{2\zeta} \tag{6.4}$$

$$\operatorname{Var}(F_T) \sim T^{2\chi} \tag{6.5}$$

In this notation, the following result is a rigorous version of the inequality

$$\chi \geqslant \tfrac{1}{2}(1 - d\zeta) \tag{6.6}$$

Proposition 6.1. Suppose that for some $z > 0$ the energy-minimizing polymer ω_T satisfies

$$\operatorname{Prob}(|\omega_T(t)| \geqslant C_1 t^z) \geqslant C_2 \qquad \text{for all} \quad t \leqslant T \tag{6.7}$$

with some $C_1 > 0$ and $C_2 \in (0, 1)$. Then, for T sufficiently large,

$$\operatorname{Var}(F_T) \geqslant C T^{1 - dz} \tag{6.8}$$

with $C > 0$, which depends only on C_1, C_2, and on the distribution $v(d\eta)$.

Proof. We shall use the first part of the inequality (2.13) with $X = F_T$ and with $\eta(t, x)$ playing the role of $\tau_1, \ldots, \tau_N$. Note that for a fixed T, F_T depends only on a finite number of these variables. For a fixed pair (t, x) we have to estimate the variance of the following function of the single random variable:

$$\bar{F}_{T,(t,x)} = \int \prod_{(s, y) \neq (t,x)} v(d\eta(s, y))\, F_T \tag{6.9}$$

For every $b > a > 0$

$$\int_a^b v(d\eta(t, x)) \frac{\partial}{\partial \eta(t, x)} \bar{F}_{T,(t,x)} = \operatorname{Prob}(\omega_T(t) = x \text{ and } \eta(t, x) \in (a, b)) \tag{6.10}$$

In order to obtain from (6.10) a lower bound on the variance of $\bar{F}$, we use the following elementary lemma (whose role is similar to that played in ref. 2 by the results of Appendix III there).

Lemma 6.2. Let ν be an absolutely continuous measure on $\mathbb{R}$ with the density bounded by K [as in (6.3)]. Then for every increasing function g on $\mathbb{R}$ the inequality

$$\int_a^b g'(\eta)\,\nu(d\eta) \geqslant M \tag{6.11}$$

with $a > b$ implies

$$\mathrm{Var}_\nu(g) \geqslant \frac{\nu((-\infty, a])\,\nu([b, \infty))}{K^2} M^2 \tag{6.12}$$

where, by definition, $\mathrm{Var}_\nu(g) \equiv \int [g(\eta) - \int g(\eta')\,\nu(d\eta')]^2\,\nu(d\eta)$.

Proof. The claim is an elementary consequence of the symmetrized expression

$$\mathrm{Var}_\nu(g) = \tfrac{1}{2} \iint \nu(d\eta)\,\nu(d\eta')\,[g(\eta) - g(\eta')]^2 \quad \blacksquare \tag{6.13}$$

Proof of Proposition 6.1. For δ which will be adjusted below (at a small enough value), let $\{a, b\}$ be the points for which

$$\nu((-\infty, a]) = \nu([b, \infty)) = \delta \tag{6.14}$$

Combining Lemma 6.2 with the $\widehat{\mathrm{Var}}$ bound of (3.5), we get

$$\begin{aligned}\mathrm{Var}(F_T) &\geqslant \frac{\delta^2}{K^2} \sum_{t=1}^{T} \sum_{|x| < C_1 t^z} \mathrm{Prob}(\omega_T(t) = x \text{ and } \eta(t, x) \in (a, b))^2 \\ &\geqslant \frac{\mathrm{const}}{T^{1+dz}} \left[\sum_{t=1}^{T} \sum_{|x| < C_1 t^z} \mathrm{Prob}(\omega_T(t) = x \text{ and } \eta(t, x) \in (a, b)) \right]^2\end{aligned} \tag{6.15}$$

where in the first step the terms with $|x| > C_1 t^z$ were omitted, and in the second step we applied the Schwarz inequality.

Now (with $I[-]$ the indicator function),

$$\begin{aligned}&\sum_{t=1}^{T} \sum_{|x| < C_1 t^z} \mathrm{Prob}(\omega_T(t) = x \text{ and } \eta(t, x) \in (a, b)) \\ &\quad\geqslant \sum_{t=1}^{T} [\mathrm{Prob}(|\omega_T(t)| < C_1 t^z) - \mathrm{Prob}(\eta(t, \omega_T(t)) \notin (a, b))] \\ &\quad\geqslant C_2 T - E\left(\sum_{t=1}^{T} I[\eta(t, \omega_T(t)) \notin (a, b)] \right) \\ &\quad\geqslant C_2/2T - T\,\mathrm{Prob}(\omega_T \text{ visits at least } C_2/2\ T \text{ “bad” sites})\end{aligned} \tag{6.16}$$

where a site (t, x) is declared "bad" if $\eta(t, x) \notin (a, b)$. Such sites are distributed independently with density 2δ, which is still to be adjusted.

It remains now to show that the probability of the "bad" event in the last term of (6.16) decays faster than T^{-1}. In fact, for δ small enough, the decay is exponential in T, as the following estimate shows. For every $\kappa > 0$

$$
\begin{aligned}
&\text{Prob(the number of bad sites in } \omega_T \text{ is larger than } \kappa T) \\
&\quad \leqslant \text{Prob(there is a nearest-neighbor path of } T \text{ steps in } \mathbb{Z} \times \mathbb{Z}^d, \\
&\qquad \text{starting from 0, containing more than } \kappa T \text{ bad sites)} \\
&\quad \leqslant (2d)^T 2^T (2\delta)^{\kappa T} \equiv e^{-\lambda T} \qquad (6.17)
\end{aligned}
$$

(the factor 2^T is a bound on the number of subsets of a given path with size exceeding κT). For δ small enough, λ is positive. Choosing $\delta < 2^{-1}(4d)^{-2/C_2}$, we find that the above bounds imply

$$\operatorname{Var}(F_T) \geqslant \text{const} \cdot \frac{T^2}{T^{1+dz}} = \text{const} \cdot T^{1-dz} \qquad (6.18)$$

which for large enough T holds with a constant depending only on C_1, C_2, and K. ▮

Remarks. (a) In order to relate the condition (6.7) to the heuristic definition of the exponent ζ, let us note that (6.7) follows (by Chebyshev inequality) if we assume the moment condition

$$\langle |\omega_T(t)|^p \rangle \leqslant A t^{pz} \qquad (6.19)$$

with some $p > 0$, $A > 0$.

(b) For $d = 1$ the exact value of ζ is believed to be 2/3. The inequality (6.1) would then imply $\chi \geqslant 1/6$, which is consistent with the relation $\chi = 2\zeta - 1$ derived heuristically in ref. 13.

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